Wuhan University Journal of Natural Sciences

2026, Vol.31 No.5, 459-470

Article ID 1007-1202(2026)05-0459-12 DOI https://doi.org/10.1051/wujns/2026315459

Cite this article: HU Xueping, WANG Ziyu. Asymptotic Normality for the Edge Frequency Polygons under Asymptotically Negatively Associated Samples[J]. *Wuhan Univ J of Nat Sci*, 2026, 31(5): 459-470.

Asymptotic Normality for the Edge Frequency Polygons under Asymptotically Negatively Associated Samples

HU Xueping, WANG Ziyu

School of Mathematics and Statistics, Anqing Normal University, Anqing 246133, Anhui, China

Abstract: The edge frequency polygon estimator, which is based on the histogram technique, has a convergence rate similar to that of the nonnegative kernel estimator and possesses the advantage of computational simplicity. In this paper, the uniformly asymptotic normality of the edge frequency polygon estimator for asymptotically negatively associated (ANA) random variables is established under suitable conditions, and the convergence rate is approximately $O\left(n^{-1/9}\right)$, which reveals that the edge frequency polygon estimator has a similar Berry-Esséen bound of the kernel estimator. Additionally, some numerical simulations are carried out to validate the theoretical results.

Key words: uniformly asymptotic normality; edge frequency polygon estimator; asymptotically negatively associated (ANA) random variables

CLC number: O212.7

0 Introduction

The study of density estimation is significant because it provides the basis for general nonparametric and semiparametric estimation. The main methods include histogram estimation, frequency polygon density estimation, kernel density estimation and nearest neighbor estimation, etc. As pointed out by Scott[1], the frequency polygon estimator has a convergence rate similar to that of the kernel density estimator and faster than that of the histogram. Moreover, the computational effort of the frequency polygon is equivalent to that of the histogram. For large bivariate datasets, the computational simplicity of the frequency polygons and the ease of determining exact equiprobable contours may outweigh the increased accuracy of a kernel density estimator.

The edge frequency polygons, initially proposed by Jones *et al*[2], serve as a piecewise linear alternative to the frequency polygons. We herein recall the notation of the edge frequency polygon. Suppose $\{X_n, n\geqslant 1\}$ is a sequence of random variables with common continuous density function $f(x)$ on the real line $\mathbb{R}$. Consider a partition $\cdots<x_{-2}<x_{-1}<x_0<x_1<x_2<\cdots$ of $\mathbb{R}$ into equal intervals, the k-th interval denotes $I_k=\left[\frac{(2k-1)b_n}{2},\ \frac{(2k+1)b_n}{2}\right)$ of length b_n, where b_n is the

Received date: 2025-11-13 © Wuhan University 2026

Foundation item: Supported by the National Social Science Foundation of China (21BTJ040)

Biography: HU Xueping, male, Professor, research direction: nonparametric large sample theory. E-mail: hxprob@163.com

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/ licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

bin width with $b_n \to 0$ as $n \to \infty$. For any $x \in \mathbb{R}$, there exists some integer $k_0(x) \doteq k_0$ such that $\left(k_0 - \frac{1}{2}\right)b_n \leqslant x < \left(k_0 + \frac{1}{2}\right)b_n$. Consider three adjacent histogram bins

$$I_{k_{-1}} = \left[\left(k_{-1} - \frac{1}{2}\right)b_n,\ \left(k_{-1} + \frac{1}{2}\right)b_n\right),\ I_{k_0} = \left[\left(k_0 - \frac{1}{2}\right)b_n,\ \left(k_0 + \frac{1}{2}\right)b_n\right) \text{ and } I_{k_1} = \left[\left(k_1 - \frac{1}{2}\right)b_n, \left(k_1 + \frac{1}{2}\right)b_n\right),$$

where $k_{-1} = k_0 - 1$ and $k_1 = k_0 + 1$. Define

$$v_{k_s} = \sum_{i=1}^{n} I\left(\left(k_s - \frac{1}{2}\right)b_n \leqslant X_i < \left(k_s + \frac{1}{2}\right)b_n\right),\ \ s = -1, 0, 1,$$

which are the numbers of the observations falling into the intervals mentioned above, respectively. The values of the histogram in these previous bins can be denoted by $f_{k_s} = v_{k_s}(nb_n)^{-1}, s = -1, 0, 1,$ respectively. Therefore, the edge frequency polygon estimator $\hat{f}_n(x)$ of the density function $f(x)$ can be defined as

$$\hat{f}_n(x) = \left(\frac{1}{2} + k_0 - \frac{x}{b_n}\right)\frac{f_{k_{-1}} + f_{k_0}}{2} + \left(\frac{1}{2} - k_0 + \frac{x}{b_n}\right)\frac{f_{k_0} + f_{k_1}}{2}, \tag{1}$$

for $x \in \left[(k_0 - \frac{1}{2})b_n,\ (k_0 + \frac{1}{2})b_n\right)$.

The edge frequency polygons offer the advantage of being as straightforward and interpretable as the ordinary frequency polygons, while simultaneously offering both theoretical and practical benefits. Given these advantages, it is of considerable significance to explore the edge frequency polygons more deeply.

The frequency polygon estimator $\widetilde{f}_n(x)$ of the density function $f(x)$ can be defined as

$$\widetilde{f}_n(x) = \left(\frac{1}{2} + k_0 - \frac{x}{b_n}\right)\dot{f}_{k_0} + \left(\frac{1}{2} - k_0 + \frac{x}{b_n}\right)\dot{f}_{k_1}, \tag{2}$$

for $x \in \left[(k_0 - \frac{1}{2})b_n,\ (k_0 + \frac{1}{2})b_n\right)$, where $\dot{f}_{k_s} = (nb_n)^{-1}\sum_{i=1}^{n} I\left((k_s - 1)b_n \leqslant X_i < k_s b_n\right),\ s = 0, 1$. Recently, some scholars have contributed fruitful research results in frequency polygons. For instance, Carbon *et al*[3] proved the asymptotic normality of the frequency polygons for random fields. Yang[4] further investigated the uniformly strong consistency of the frequency polygons for strong mixing samples. Xing *et al*[5] proved the uniformly strong consistency of the estimator and obtained a convergence rate. Wen and Xing[6] examined the asymptotic normality of the frequency polygons for ψ-mixing samples, and the convergence rate is approximately $O(n^{-1/6})$. Xing and Yang[7] extended the uniformly strong consistency of the estimator for negatively associated samples. Xing *et al*[8] explored the uniformly strong consistency and Berry-Esséen bound of the frequency polygons for α-mixing samples. Kang *et al*[9] studied the uniformly strong consistency of the estimator for ψ-mixing samples. Xi *et al*[10] studied the uniformly strong consistency and the convergence rate of asymptotic normality for the edge frequency polygon estimator. Younso[11] examined the strong consistency of the frequency polygon estimators for strongly mixing sequences of real variables. Wu and Wang[12] established the Berry-Esséen bound of the frequency polygon estimation with ρ-mixing samples. Additionally, Wang *et al*[13] investigated the uniformly complete consistency of the frequency polygon estimation under *m*-extended negatively dependent (*m*-END) samples. Chao and Hu[14] obtained the convergence rate of asymptotic normality of the frequency polygon estimator for widely orthant dependent (WOD) random variables.

To the best of our knowledge, the existing literature does not provide information on the rate of asymptotic normality for the edge frequency polygon estimator $\hat{f}_n(x)$ under asymptotically negatively associated (ANA) samples. Inspired by the literature above, we aim to establish the convergence rates of asymptotic normality under suitable conditions in this paper. Finally, we conduct numerical simulations to provide support for the theoretical results.

Now let us recall some concepts of dependence structures. The concept of ρ^*-mixing dependent random variables was introduced by Bradley[15] as follows.

Definition 1 A sequence $\{X_n,\ n \geqslant 1\}$ of random variables is called ρ^*-mixing if

$$\rho^*(s) = \sup\{\rho(S,\ T);\ S,\ T \subset \mathbb{N},\ \operatorname{dist}(S, T) \geqslant s\} \to 0,\ \text{as } s \to \infty,$$

$$\rho(S,T)=\sup\left\{\frac{|E(XY)-EX\cdot EY|}{\sqrt{\mathrm{Var}(X)\cdot\mathrm{Var}(Y)}}: X\in L_2\big(\sigma(X_i,i\in S)\big),\ Y\in L_2\big(\sigma(X_j,j\in T)\big)\right\},$$

where $\mathrm{dist}(S,T)=\min\limits_{i\in S,\,j\in T}|j-i|$.

Zhang and Wang[16] introduced the following concept of ANA random variables.

Definition 2 A sequence $\{X_n, n\geqslant 1\}$ of random variables is called ANA (or ρ^--mixing) if

$$\rho^-(s)=\sup\{\rho^-(S,T);\ S,\ T\subset\mathbb{N},\ \mathrm{dist}(S,T)\geqslant s\}\to 0,\ \text{as}\ s\to\infty,$$

where $\rho^-(S,T)=0\vee\sup\left\{\dfrac{\mathrm{Cov}\big(f_1(X_i,i\in S),f_2(X_j,j\in T)\big)}{\sqrt{\mathrm{Var}\big(f_1(X_i,i\in S)\big)\cdot\mathrm{Var}\big(f_2(X_j,j\in T)\big)}}: f_1,f_2\in\mathcal{C}\right\}$, and $\mathcal{C}$ is the set of nondecreasing functions.

An array of random variables $\{X_{ni},\ 1\leqslant i\leqslant n,\ n\geqslant 1\}$ is said to be rowwise ANA if for every $n\geqslant 1$, $\{X_{ni},\ 1\leqslant i\leqslant n\}$ are ANA.

Remark 1 It is obviously seen that $\rho^-(s)\leqslant\rho^*(s)$ and ANA implies negatively associated (NA) random variables if and only if $\rho^-(s)=0$ for $s\geqslant 1$. Zhang and Wang[16] pointed out that ρ^*-mixing and NA are both ANA; however, the converse is not always true. They presented an example of an ANA sequence which is neither NA nor ρ^*-mixing and proved that the increasing functions defined on disjoint subsets of an ANA sequence $\{X_n, n\geqslant 1\}$ with mixing coefficients $\rho^-(s)$ are also ANA with mixing coefficients not greater than $\rho^-(s)$. Hence, the investigation of the limit properties for ANA random variables is of great interest.

Since the concept of ANA random variables was introduced by Zhang and Wang[16], many interesting theoretical results have been established. Zhang[17] obtained the central limit theorems. Tang *et al*[18] explored the asymptotic normality of a wavelet estimator for asymptotically negatively associated errors. Wu *et al*[19] established a general result on complete moment convergence and the Marcinkiewicz-Zygmund-type strong law of large numbers for weighted sums of m-ANA random variables, and so on.

In this work, we will study the asymptotic normality of the edge frequency polygon estimation for ANA samples. The structure of the paper is as follows: The main results are presented in Section 1. Section 2 illustrates some lemmas and proofs of the main results. Some numerical simulations are carried out in Section 3.

Throughout this paper, the symbols C, and C_p denote positive constants, which do not depend on n and may take different values in various positions. The bin width and the density function are denoted by b_n and $f(x)$, respectively. All limits are taken as the sample size $n\to\infty$, unless specified otherwise. Let $\lfloor x\rfloor$ be the integer part of x and $I(A)$ represents the indicator function of a set A, $\Phi(u)$ is the standard normal distribution function. Set $\|X\|_r=\big(E|X|^r\big)^{1/r}$, $\|X\|_{2,1}=\int_0^\infty P^{1/2}(|X|\geqslant x)\mathrm{d}x$, $x^+=xI(x>0)$, $x^-=-xI(x<0)$.

1 Assumptions and Main Results

1.1 Basic Assumptions

For the presentation of our main results, the following assumptions are indispensable.

(A1) Let $\{X, X_n,\ n\geqslant 1\}$ be a sequence of identically distributed ANA random variables with $\rho^-(s)\leqslant C(\log s)^{-1-\delta}$ for some $C>0$ and $\delta>0$. Let $u(n)=\sum\limits_{j=n}^{\infty}\rho^-(j)\to 0$, as $n\to\infty$.

(A2) The continuous density function $f(x)$ of X satisfies $f(x)\leqslant M$ for $x\in\mathbb{R}$ and some $M>0$. The joint density function $f_{i,j}(x,y)$ of (X_i,X_j) exists and satisfies $\sup\limits_{(x,y)\,\in\,\mathbb{R}\times\mathbb{R}}\big|f_{i,j}(x,y)-f(x)f(y)\big|\leqslant M_1$, for each $i<j$ and some $M_1>0$.

(A3) Set $\sigma_n^2(x):=\mathrm{Var}(\hat{f}_n(x))>0$, let $p\doteq p_n$ and $q\doteq q_n$ be positive integers tending to ∞, such that $p+q\leqslant n$, $qp^{-1}\leqslant C<\infty$. The bin width $b_n\to 0$ and $pb_n\to\infty$, as $n\to\infty$.

Let $\gamma_{in}\to 0$, $(i=1,2)$, $v(n)u^{-2/3}(q)\to 0$, as $n\to\infty$, where $\gamma_{1n}=q/p$, $\gamma_{2n}=p/n$, $v(n)=np^{-1}\rho^-(q)$.

Remark 2 The assumptions above are very general. Assumption (A1) is a regular condition, for details, please

refer to Refs. [16, 18]. For assumption (A2) one can refer to Refs. [8, 19]. However, assumption (A3) can be satisfied easily by choosing appropriate p_n, q_n, b_n and $\rho^-(n)$.

1.2 Main Results

The following result is about the Berry-Esséen bound of the edge frequency polygon estimator for ANA samples.

Theorem 1 Suppose that assumptions (A1)-(A3) are satisfied. For any $x \in \mathbb{R}$ such that $f(x)>0$, we have

$$\sup_{-\infty<u<\infty} \left| F_n(u) - \Phi(u) \right| \leqslant C\left(\gamma_{1n}^{1/3} + \gamma_{2n}^{1/3} + u^{1/3}(q) + v(n)u^{-2/3}(q)\right), \tag{3}$$

where $F_n(u) = P\left(S_n \leqslant u\right)$ and $S_n = \sigma_n^{-1}(x)\left[\hat{f}_n(x) - E\hat{f}_n(x)\right]$.

Corollary 1 Suppose that assumptions (A1)-(A2) hold, take $p = \lfloor n^{2/3} \rfloor$, $q = \lfloor n^{1/3} \rfloor$, $b_n = O\left(n^{-1/3}\right)$, $\rho^-(n) = O\left(n^{-2}\right)$. For some $r>2$, we obtain $\sup_{-\infty<u<\infty} \left| F_n(u) - \Phi(u) \right| = O\left(n^{-1/9}\right)$.

Remark 3 Xing *et al*[8] studied the uniformly asymptotic normality of the frequency polygon for α-mixing samples, and obtained the convergence rate $O\left(n^{-1/11}\right)$. Wen and Xing[6] has derived the convergence rate to be close to $O\left(n^{-1/6}\right)$ under ψ-mixing samples.

The following results can be obtained with some additional assumptions.

Theorem 2 Suppose that assumptions (A1)-(A3) are satisfied. If $nb_n^3 \to 0$ as $n \to \infty$, and $f(x)$ satisfies the local Lipschitz condition around x with $f(x)>0$, then

$$\sup_{-\infty<u<\infty} \left| P\left(\widetilde{S}_n \leqslant u\right) - \Phi(u) \right| \leqslant \left(\gamma_{1n}^{1/3} + \gamma_{2n}^{1/3} + u^{1/3}(q) + v(n)u^{-2/3}(q) + \left(nb_n^3\right)^{1/2}\right), \tag{4}$$

where $\widetilde{S}_n = \sigma_n^{-1}(x)\left[\hat{f}_n(x) - f(x)\right]$.

According to Theorem 2 the asymptotic normality and the convergence rate of estimator $\hat{f}_n(x)$ can be obtained under ANA samples.

Corollary 2 Suppose that (A1) and (A2) hold with $\rho^-(n) = O(n^{-2})$. Let $b_n = O\left(n^{-\alpha}\right)$ and $11/27 \leqslant \alpha < 2/3$. If $f(\cdot)$ satisfies the local Lipschitz condition around x with $f(x)>0$, we have $\sup_{-\infty<u<\infty} \left| P\left(\widetilde{S}_n \leqslant u\right) - \Phi(u) \right| = O\left(n^{-1/9}\right)$.

Corollary 3 Suppose that (A1)-(A2) hold. If $pb_n \to \infty$ and $nb_n^3 \to 0$ as $n \to \infty$, and $f(\cdot)$ satisfies the local Lipschitz condition around x with $f(x)>0$, then $\widetilde{S}_n \xrightarrow{D} N(0,1)$, as $n \to \infty$, where D denotes the convergence in distribution.

Remark 4 (i) For the frequency polygon estimator of the density function $f(x)$, Xing *et al*[8] obtained the rate $O\left(n^{-1/11}\right)$ of $P\left(\dfrac{\widetilde{f}_n(x) - E\widetilde{f}_n(x)}{\sigma_n(x)} \leqslant u\right)$ under stationary α-mixing samples. Additionally, Wu and Wang[12] studied the results of the convergence of $P\left(\dfrac{\widetilde{f}_n(x) - f(x)}{\sigma_n(x)} \leqslant u\right)$ to the standard normal distribution, obtained a better rate $O\left(n^{-1/9}\right)$ with $\rho(n) = O\left(n^{-4/3}\right)$.

(ii) In this paper, our results extend to the edge frequency polygon estimator and establish the asymptotic normality of the distribution function of $\widetilde{S}_n$ for ANA samples. Furthermore, we also obtain the convergence rate $O\left(n^{-1/9}\right)$ of $P\left(\widetilde{S}_n \leqslant u\right)$ approximating to the standard normal distribution.

2 Proofs of Theorems

In this section, we prove the Berry-Esséen bound for the edge frequency polygon estimator based on ANA samples. For convenient writing, we firstly provide the following notations. For each $1 \leqslant i \leqslant n, s = -1, 0, 1,$ denote

$$Y_{i,k_s} = I\left\{\left(k_s - 1/2\right)b_n \leqslant X_i < (k_s + 1/2)b_n\right\}, \quad e_{i,k_s} := Y_{i,k_s} - EY_{i,k_s},$$

$$\eta_i(x) := \left(\frac{1}{2} + k_0 - \frac{x}{b_n}\right)\frac{e_{i,k_{-1}} + e_{i,k_0}}{2} + \left(\frac{1}{2} - k_0 + \frac{x}{b_n}\right)\frac{e_{i,k_0} + e_{i,k_1}}{2},$$

$$Z_{ni} := \left(nb_n\sigma_n(x)\right)^{-1}\eta_i(x),$$

then $S_n=\sigma_n^{-1}(x)\left[\hat{f}_n(x)-E\hat{f}_n(x)\right]=\sum_{i=1}^{n}Z_{ni}$.

Let $k:=k_n=\lfloor n/(p+q)\rfloor$, by employing Bernstein's big-block and small-block procedures, S_n can be decomposed into

$$S_n=S'_n+S''_n+S'''_n, \tag{5}$$

where

$$S'_n=\sum_{m=1}^{k}y'_{nm},\ S''_n=\sum_{m=1}^{k}y''_{nm},\ S'''_n=y'''_n,$$

and

$$y'_{nm}=\sum_{i=k_m}^{k_m+p-1}Z_{ni},\ y''_{nm}=\sum_{i=l_m}^{l_m+q-1}Z_{ni},\ y'''_n=\sum_{i=k(p+q)+1}^{n}Z_{ni},$$

$$k_m=(m-1)(p+q)+1,\quad l_m=(m-1)(p+q)+p+1,\quad m=1,2,\cdots,k.$$

To derive the main results, we present the following helpful lemmas. The first one is a basic property of ANA random variables, which was presented by Zhang and Wang[16].

Lemma 1 Let $\{X_n,\ n\geqslant 1\}$ be a sequence of ANA random variables with mixing coefficients $\rho^-(s)$, if $\{f_n, n\geqslant 1\}$ are all nondecreasing (or nonincreasing) functions, then random variables $\{f_n(X_n),\ n\geqslant 1\}$ are still ANA with mixing coefficients not greater than $\rho^-(s)$.

The following lemma which can be seen in Ref. [16] is the Rosenthal-type inequality for ANA random variables.

Lemma 2 Let $\{X_n, n\geqslant 1\}$ be a sequence of ANA random variables with $EX_n=0$ and $\|X_n\|_p<\infty$ for some $p\geqslant 2$ and all n. For any $k\geqslant 0,\ n\geqslant 1$, set $S_n(k)=\sum_{i=1}^{n}X_{k+i}$. Assume that $\rho^-(s)\leqslant C(\log s)^{-1-\delta}$ for some $C>0$ and $\delta>0$. Then there exists a positive constant $C(p)$ depending only on p and $\rho^-(\cdot)$ such that

$$E\left(\max_{1\leqslant m\leqslant n}|S_m(k)|^p\right)\leqslant C(p)\left\{n\max_{1\leqslant m\leqslant n}E|X_{k+m}|^p+\left(n\max_{1\leqslant m\leqslant n}E(X_{k+m}^2)\right)^{p/2}\right\}.$$

The next lemma comes from Zhang[17].

Lemma 3 Let $p,q>1$ with $1/p+1/q=1$. Suppose $X=f(X_i;i\in S),\ Y=g(X_i;i\in T)$ with $\|X\|_p<\infty$ and $\|Y\|_q<\infty$, where $S,\ T\subset\mathbb{N}$ with $S\bigcap T=\varnothing$ and $f,\ g\in\mathcal{C}$, $\mathcal{C}$ is a class of nondecreasing (or nonincreasing) functions. Then

$$|\mathrm{Cov}(X,Y)|\leqslant 6\left(\rho^-(S,T)\right)^{(2/p)\wedge(2/q)}\|X\|_p\|Y\|_q. \tag{6}$$

Lemma 4 Under assumptions (A1) and (A2), we have

$$nb_n\sigma_n^2(x)=\left(\frac{1}{2}+k_0-\frac{x}{b_n}\right)^2\frac{p_{-1}(1-p_{-1})+p_0(1-p_0)-2p_{-1}p_0}{4b_n}+\left(\frac{1}{2}-k_0+\frac{x}{b_n}\right)^2\frac{p_0(1-p_0)+p_1(1-p_1)-2p_0p_1}{4b_n}$$
$$-\left(\frac{1}{2}+k_0-\frac{x}{b_n}\right)\left(\frac{1}{2}-k_0+\frac{x}{b_n}\right)\frac{p_{-1}p_0+p_0p_1+p_{-1}p_1-p_0(1-p_0)}{2b_n}+o(1), \tag{7}$$

where $p_s=\int_{(k_s-1/2)b_n}^{(k_s+1/2)b_n}f(x)\mathrm{d}x$ for $s=-1,0,1$. Furthermore, if $f(x)$ is continuous at x, we have

$$\liminf_{n\to\infty}nb_n\sigma_n^2(x)\geqslant\frac{f(x)}{4}.$$

Proof Similar to the proof of Lemma 4.1 in Ref. [12], we can obtain (7) immediately.

Lemma 5 Suppose that assumptions (A1)-(A3) hold, we obtain

$$E(S''_n)^2\leqslant C\gamma_{1n},\ E(S'''_n)^2\leqslant C\gamma_{2n}. \tag{8}$$

Proof According to C_r-inequality, Jensen's inequality and $f(x)\leqslant M$, it follows that for any $r>1$,

$$\max_{1\leqslant i\leqslant n}E|\eta_i(x)|^r\leqslant 2^{r-1}\max_{1\leqslant i\leqslant n}E\left|\frac{e_{i,k_{-1}}(x)+e_{i,k_0}(x)}{2}\right|^r+2^{r-1}\max_{1\leqslant i\leqslant n}E\left|\frac{e_{i,k_1}(x)+e_{i,k_0}(x)}{2}\right|^r$$
$$\leqslant C\left(\int_{(k_{-1}-1/2)b_n}^{(k_{-1}+1/2)b_n}f(t)\mathrm{d}t+\int_{(k_0-1/2)b_n}^{(k_0+1/2)b_n}f(t)\mathrm{d}t+\int_{(k_1-1/2)b_n}^{(k_1+1/2)b_n}f(t)\mathrm{d}t\right)\leqslant CMb_n, \tag{9}$$

then we can deduce that by (9) and (7)

$$E\left|\eta_i(x)\right|^2 \leqslant Cb_n,\ EZ_{ni}^2 = E\left|\left(nb_n\sigma_n(x)\right)^{-1}\eta_i(x)\right|^2 \leqslant \frac{C}{n}.$$

By Lemma 2, Lemma 4 and (9), we easily see that

$$E(S''_n)^2 = E\left(\sum_{m=1}^{k} y''_{nm}\right)^2 = E\left(\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1} Z_{ni}\right)^2 \leqslant C\left(nb_n\sigma_n(x)\right)^{-2}\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1}\max_{1\leqslant i\leqslant n}E|\eta_i(x)|^2 \leqslant C\frac{kq}{n^2b_n\sigma_n^2(x)} \leqslant \frac{Cq}{p} = C\gamma_{1n},$$

and $E\left(S'''_n\right)^2 = E\left(\sum_{i=k(p+q)+1}^{n} Z_{ni}\right)^2 \leqslant C\sum_{i=k(p+q)+1}^{n}\frac{1}{\left(nb_n^{\frac{1}{2}}\sigma_n(x)\right)^2} \leqslant C\frac{p}{n^2b_n\sigma_n^2(x)} \leqslant \frac{Cp}{n} = C\gamma_{2n}.$

Hence, (8) holds, and we get by Markov's inequality $P\left(\left|S''_n \geqslant \gamma_{1n}^{\frac{1}{3}}\right|\right) \leqslant C\gamma_{1n}^{\frac{1}{3}},\ P\left(\left|S'''_n \geqslant \gamma_{2n}^{\frac{1}{3}}\right|\right) \leqslant C\gamma_{2n}^{\frac{1}{3}}.$

Lemma 6 Suppose that assumptions of Theorem 1 are satisfied, we have

$$\left|s_n^2 - 1\right| \leqslant C\left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + u(q)\right), \tag{10}$$

where $s_n^2 = \sum_{m=1}^{k}\mathrm{Var}\left(y'_{nm}\right)$.

Proof Denote $\Gamma_n = \sum_{1\leqslant i\leqslant j\leqslant k}\mathrm{Cov}\left(y'_{ni}, y'_{nj}\right)$. Obviously, we obtain

$$s_n^2 = E\left(S'_n\right)^2 - 2\Gamma_n. \tag{11}$$

Noting that $E\left(S_n\right)^2 = 1$, we easily see that $E\left(S'_n\right)^2 = 1 - 2E\left[S_n\left(S''_n + S'''_n\right)\right] + E\left(S''_n + S'''_n\right)^2$, which combining with C_r-inequality and (8), we can derive

$$\begin{aligned}\left|E(S'_n)^2 - 1\right| &\leqslant 2E\left(S''_n\right)^2 + 2E\left(S'''_n\right)^2 + 2\left[E\left(S_n^2\right)\right]^{\frac{1}{2}}\left[E\left(S''_n\right)^2\right]^{\frac{1}{2}} + 2\left[E\left(S_n^2\right)\right]^{\frac{1}{2}}\left[E\left(S'''_n\right)^2\right]^{\frac{1}{2}}\\ &\leqslant 2\left(\gamma_{1n} + \gamma_{2n}\right) + 2\gamma_{1n}^{\frac{1}{2}} + 2\gamma_{2n}^{\frac{1}{2}} \leqslant C\left(\gamma_{1n}^{\frac{1}{2}} + \gamma_{2n}^{\frac{1}{2}}\right).\end{aligned} \tag{12}$$

On the other hand, for $x \in [(k_0 - 1/2)b_n,\ (k_0 + 1/2)b_n)$, noting that $0 < 1/2 + k_0 - x/b_n \leqslant 1$ and $0 < 1/2 - k_0 + x/b_n \leqslant 1$, it follows that

$$\begin{aligned}\left|\Gamma_n\right| &\leqslant \sum_{1\leqslant i<j\leqslant k}\left|\mathrm{Cov}\left(y'_{ni}, y'_{nj}\right)\right| \leqslant \sum_{1\leqslant i<j\leqslant k}\sum_{v=k_i}^{k_i+p-1}\sum_{t=k_j}^{k_j+p-1}\left|\mathrm{Cov}\left(\mathrm{Z}_{nv}, Z_{nt}\right)\right| \leqslant \sum_{1\leqslant i<j\leqslant k}\sum_{v=k_i}^{k_i+p-1}\sum_{t=k_j}^{k_j+p-1}\frac{1}{\left(nb_n\sigma_n(x)\right)^2}\left|\mathrm{Cov}\left(\eta_v, \eta_t\right)\right|\\ &\leqslant C\left(nb_n\sigma_n(x)\right)^{-2}\sum_{1\leqslant i<j\leqslant k}\sum_{v=k_i}^{k_i+p-1}\sum_{t=k_j}^{k_j+p-1} \times \left\{\left|\left(\frac{1}{2} + k_0 - \frac{x}{b_n}\right)^2\mathrm{Cov}\left(\frac{e_{v,k_{-1}}(x) + e_{v,k_0}(x)}{2}, \frac{e_{t,k_{-1}}(x) + e_{t,k_0}(x)}{2}\right)\right|\right.\\ &\quad + \left|\left(\frac{1}{2} - k_0 + \frac{x}{b_n}\right)^2\mathrm{Cov}\left(\frac{e_{v,k_0}(x) + e_{v,k_1}(x)}{2}, \frac{e_{t,k_0}(x) + e_{t,k_1}(x)}{2}\right)\right|\\ &\quad + \left|\left(\frac{1}{2} + k_0 - \frac{x}{b_n}\right)\left(\frac{1}{2} - k_0 + \frac{x}{b_n}\right)\mathrm{Cov}\left(\frac{e_{v,k_{-1}}(x) + e_{v,k_0}(x)}{2}, \frac{e_{t,k_0}(x) + e_{t,k_1}(x)}{2}\right)\right|\\ &\quad \left.+ \left|\left(\frac{1}{2} + k_0 - \frac{x}{b_n}\right)\left(\frac{1}{2} - k_0 + \frac{x}{b_n}\right)\mathrm{Cov}\left(\frac{e_{v,k_0}(x) + e_{v,k_1}(x)}{2}, \frac{e_{t,k_{-1}}(x) + e_{t,k_0}(x)}{2}\right)\right|\right\}\\ &\leqslant C\left(nb_n\sigma_n(x)\right)^{-2}\sum_{1\leqslant i<j\leqslant k}\sum_{v=k_i}^{k_i+p-1}\sum_{t=k_j}^{k_j+p-1}\left\{\left|q_{1n}\right| + \left|q_{2n}\right| + \left|q_{3n}\right| + \left|q_{4n}\right|\right\},\end{aligned}$$

where

$$q_{1n} = \mathrm{Cov}\left(\frac{e_{v,k_{-1}}(x) + e_{v,k_0}(x)}{2},\ \frac{e_{t,k_{-1}}(x) + e_{t,k_0}(x)}{2}\right),\ \ q_{2n} = \mathrm{Cov}\left(\frac{e_{v,k_0}(x) + e_{v,k_1}(x)}{2},\ \frac{e_{t,k_0}(x) + e_{t,k_1}(x)}{2}\right),$$

$$q_{3n} = \mathrm{Cov}\left(\frac{e_{v,k_{-1}}(x) + e_{v,k_0}(x)}{2},\ \frac{e_{t,k_0}(x) + e_{t,k_1}(x)}{2}\right),\ \ q_{4n} = \mathrm{Cov}\left(\frac{e_{v,k_0}(x) + e_{v,k_1}(x)}{2},\ \frac{e_{t,k_{-1}}(x) + e_{t,k_0}(x)}{2}\right),$$

for a given k_s, $s=-1,0,1$ and $i=1,2,\cdots,n$, by Lemma 1, it follows that $\{e_{i,k_s}, 1\leqslant i\leqslant n\}$ are also ANA sequences with $|e_{i,k_s}|\leqslant 2$ and $Ee_{i,k_s}=0$, which together with Lemma 3, assumption (A1), yields the following upper bound

$$|q_{1n}|=\frac{1}{4}\sum_{w=-1,0}\sum_{m=-1,0}\left|\mathrm{Cov}\left(e_{v,k_w},e_{t,k_m}\right)\right|\leqslant\frac{3}{2}\sum_{w,m=-1,0}\rho^{-}(t-v)\|e_{v,k_w}\|_2\|e_{t,k_m}\|_2\leqslant CM\rho^{-}(t-v)b_n,$$

similarly, we can conclude that $|q_{in}|\leqslant CM\rho^{-}(t-v)b_n$, $i=2,3,4$.

Hence, applying Lemma 4 it can be deduced that

$$|\Gamma_n|\leqslant C\left(nb_n\sigma_n(x)\right)^{-2}kpb_n\sum_{j=q}^{\infty}\rho^{-}(j)\leqslant Cu(q),\tag{13}$$

combined with (11), (12) and (13) , the proof is completed.

Assume that $\{\eta_{nm},\ 1\leqslant m\leqslant k\}$ are independent random variables with the same distribution as that of y'_{nm} for $m=1,2,\cdots,k$. We define $T_n=\sum_{m=1}^{k}\eta_{nm}$, $B_n=\sum_{m=1}^{k}\mathrm{Var}(\eta_{nm})$, let $\widetilde{F}_n(u)$, $G_n(u)$ and $\widetilde{G}_n(u)$ be the distributions of S'_n, $\frac{T_n}{\sqrt{B_n}}$ and T_n, respectively.

Lemma 7 Under the assumptions of Theorem 1, for some $0<\delta\leqslant 1$, we get $\sup_u|G_n(u)-\Phi(u)|\leqslant C\gamma_{2n}^{\delta/2}$.

Proof Noting that $|Z_{ni}|\leqslant\frac{e_{i,k_{-1}}(x)+2e_{i,k_0}(x)+e_{i,k_1}(x)}{2nb_n\sigma_n(x)}$ and $f(x)\leqslant M$. For some $0<\delta\leqslant 1$, by Lemma 2, Lemma 4 and $pb_n\to\infty$, for large enough n we have

$$\begin{aligned}\sum_{m=1}^{k}E|\eta_{nm}|^{2+\delta}&=\sum_{m=1}^{k}E|y'_{nm}|^{2+\delta}\leqslant Ck\left(nb_n\sigma_n(x)\right)^{-2-\delta}\left\{p\max_{1\leqslant i\leqslant n}E|\eta_i|^{2+\delta}+\left(p\max_{1\leqslant i\leqslant n}E|\eta_i|^{2}\right)^{(2+\delta)/2}\right\}\\&\leqslant C\left\{\frac{pkb_n}{\left(nb_n\right)^{1+\delta/2}}+k\left(\frac{pb_n}{nb_n}\right)^{1+\delta/2}\right\}=C\left\{\left(\frac{p}{n}\right)^{\delta/2}+\left(\frac{1}{nb_n}\right)^{\delta/2}\right\}\leqslant C\gamma_{2n}^{\delta/2}.\end{aligned}$$

Furthermore, it follows from Lemma 6 that $B_n=s_n^2\to 1$, as $n\to\infty$. Applying Berry-Esséen inequality (see Ref. [20], Theorem 5.7), for $0<\delta\leqslant 1$, we get

$$\sup_u|G_n(u)-\Phi(u)|\leqslant C\frac{1}{B_n^{1+\delta/2}}\sum_{m=1}^{k}E|\eta_{nm}|^{2+\delta}\leqslant C\gamma_{2n}^{\delta/2}.$$

The proof of Lemma 7 is completed.

The following lemma was proved in Ref. [18], which plays a key role to prove the main results in this paper.

Lemma 8 Let $\{X_n,\ n\geqslant 1\}$ be a sequence of ANA random variables with finite variances, p and q be two integers. Let $\eta_l=\sum_{i=(l-1)(p+q)+1}^{(l-1)(p+q)+p}X_i$ for $1\leqslant l\leqslant k$. Then, for any finite real numbers r_l, $l=1,\cdots,k$,

$$\left|E\left(\exp\left(\mathrm{i}\sum_{l=1}^{k}r_l\eta_l\right)\right)-\prod_{l=1}^{k}E\left(\exp\left(\mathrm{i}r_l\eta_l\right)\right)\right|\leqslant 4\sum_{\substack{j\neq l\\ j,l=1}}^{k}|r_jr_l|\left\{-\mathrm{Cov}(\eta_j,\eta_l)+8\rho^{-}(q)\|\eta_j\|_{2,1}\|\eta_l\|_{2,1}\right\}.$$

Lemma 9 Under the conditions of Theorem 1, we have $\sup_u\left|\widetilde{F}_n(u)-\widetilde{G}_n(u)\right|\leqslant C\left(\gamma_{2n}^{\delta/2}+u^{1/3}(q)+v(n)u^{-2/3}(q)\right)$.

Proof Denote the characteristic functions of S'_n and T_n by $\varphi(t)$ and $\psi(t)$, respectively. By Berry-Esséen inequality (see Ref. [20], Theorem 5.3), then for any $T>0$,

$$\sup_u\left|\widetilde{F}_n(u)-\widetilde{G}_n(u)\right|\leqslant\int_{-T}^{T}\left|\frac{\varphi(t)-\psi(t)}{t}\right|\mathrm{d}t+T\sup_u\int_{|y|\leqslant c/T}\left|\widetilde{G}_n(u+y)-\widetilde{G}_n(u)\right|\mathrm{d}y:=I_1+I_2.$$

Noting that $\psi(t)=E\left(\exp(\mathrm{i}tT_n)\right)=\prod_{m=1}^{k}E\left(\exp\left(\mathrm{i}t\eta_{nm}\right)\right)=\prod_{m=1}^{k}E\left(\exp\left(\mathrm{i}ty'_{nm}\right)\right)$, applying Lemma 8, and we derive that

$$\begin{aligned}|\varphi(t)-\psi(t)|&=\left|E\left(\exp\left(\mathrm{i}t\sum_{m=1}^{k}\eta_{nm}\right)\right)-\prod_{m=1}^{k}E\left(\exp\left(\mathrm{i}ty'_{nm}\right)\right)\right|\\&\leqslant 4t^2\left\{\sum_{1\leqslant i<j\leqslant k}|\mathrm{Cov}\left(y'_{ni},y'_{nj}\right)|+8\sum_{1\leqslant i<j\leqslant k}\rho^{-}(q)\|y'_{ni}\|_{2,1}\|y'_{nj}\|_{2,1}\right\}\leqslant Ct^2\left\{I_{11}+I_{12}\right\},\end{aligned}$$

where $I_{11}=\sum_{1\leqslant i<j\leqslant k}|\operatorname{Cov}\left(y'_{ni},y'_{nj}\right)|$, $I_{12}=8\sum_{1\leqslant i<j\leqslant k}\rho^{-}(q)\|y'_{ni}\|_{2,1}\|y'_{nj}\|_{2,1}$.

For I_{12}, noting that $\|X\|_{2,1}\leqslant\frac{r}{r-2}\|X\|_r$ for any $r>2$ (cf. Ledoux[21]), by C_r-inequality, Lemma 2, Lemma 4 and (9), it follows that

$$I_{12}\leqslant C\rho^{-}(q)\left\{\sum_{m=1}^{k}\left\|y'_{nm}\right\|_{\tau}\right\}^{2}\leqslant Ck\rho^{-}(q)\sum_{m=1}^{k}\left\|y'_{nm}\right\|_{r}^{2}\leqslant Ck\rho^{-}(q)\left(nb_n\sigma_n(x)\right)^{-2}\sum_{m=1}^{k}\left\{p\max_{1\leqslant i\leqslant n}E|\eta_i|^r+\left(p\max_{1\leqslant i\leqslant n}E|\eta_i|^2\right)^{(r/2)}\right\}^{2/r}$$
$$\leqslant Ck\rho^{-}(q)\left\{1+(pb_n)^{1-r/2}\right\}^{2/r}\leqslant Cnp^{-1}\rho^{-}(q)=Cv(n),$$

which together with (13), we have $\left|\varphi(t)-\psi(t)\right|\leqslant Ct^2\left(\left(u(q)+v(n)\right)\right)$, which implies that

$$I_1=\int_{-T}^{T}\left|\frac{\varphi(t)-\psi(t)}{t}\right|\mathrm{d}t\leqslant C\left(\left(u(q)+v(n)\right)T^2.\right. \tag{14}$$

Moreover, noting that $\widetilde{G_n}(u)=G_u(u/s_n)$ we obtain by Lemma 7 and Lemma 6

$$\sup_u\left|\widetilde{G_n}(u+y)-\widetilde{G_n}(u)\right|=\sup_u\left|G_n\left((u+y)/s_n\right)-G_n\left(u/s_n\right)\right|$$
$$\leqslant\sup_u\left(G_n\left((u+y)/s_n\right)-\Phi\left((u+y)/s_n\right)\right)+\sup_u\left|\Phi\left((u+y)/s_n\right)-\Phi(u/s_n)+\sup_u\left|G_n\left(u/s_n\right)-\Phi\left(u/s_n\right)\right|\right|$$
$$\leqslant 2\sup_u\left|G_n(u)-\Phi(u)\right|+\sup_u\left|\Phi\left((u+y)/s_n\right)-\Phi\left(u/s_n\right)\right|\leqslant C\left(\gamma_{2n}^{\delta/2}+|y|\right). \tag{15}$$

From (15) it follows that

$$I_2\leqslant CT\int_{|y|\leqslant c/T}\left(\gamma_{2n}^{\delta/2}+|y|\right)\mathrm{d}y\leqslant C\left(\gamma_{2n}^{\delta/2}+1/T\right), \tag{16}$$

combing (14) and (16), we deduce that

$$\sup_u\left|\widetilde{F_n}(u)-\widetilde{G_n}(u)\right|\leqslant C\left\{(u(q)+v(n))T^2+\gamma_{2n}^{\delta/2}+\frac{1}{T}\right\},$$

by choosing $T=u^{-1/3}(q)$, the proof of Lemma 9 is completed.

The last one comes from Ref. [22].

Lemma 10 Assume that $\{\zeta_n,n\geqslant 1\}$ and $\{\eta_n,n\geqslant 1\}$ are two random variable sequences, $\{\gamma_n,n\geqslant 1\}$ is a positive constant sequence, and $\gamma_n\to 0$, if $\sup_u|F_{\zeta_n}(u)-\Phi(u)|\leqslant C\gamma_n$, then for any $\varepsilon>0$,

$$\sup_u\left|F_{\zeta_n+\eta_n}(u)-\Phi(u)\right|\leqslant C\left(\gamma_n+\varepsilon+P(|\eta_n|\geqslant\varepsilon)\right). \tag{17}$$

Now we give the proof of the main results as follows.

Proof of Theorem 1 It is obvious that

$$\sup_u\left|\widetilde{F_n}(u)-\Phi(u)\right|\leqslant\sup_u\left|\widetilde{F_n}(u)-\widetilde{G_n}(u)+\widetilde{G_n}(u)-\Phi(u/\sqrt{B_n})+\Phi(u/\sqrt{B_n})-\Phi(u)\right|$$
$$\leqslant\sup_u\left|\widetilde{F_n}(u)-\widetilde{G_n}(u)\right|+\sup_u\left|\widetilde{G_n}(u)-\Phi(u/\sqrt{B_n})\right|+\sup_u\left|\Phi(u/\sqrt{B_n})-\Phi(u)\right|$$
$$:=J_{1n}+J_{2n}+J_{3n}.$$

Based on the results from Lemmas 9, 7 and 6, we can conclude

$$J_{1n}=\sup_u\left|\widetilde{F_n}(u)-\widetilde{G_n}(u)\right|\leqslant C\left(\gamma_{2n}^{\delta/2}+u^{1/3}(q)+v(n)u^{-2/3}(q)\right),$$

$$J_{2n}=\sup_u\left|\widetilde{G_n}(u)-\Phi(u/\sqrt{B_n})\right|=\sup_u\left|G_n(u)-\Phi(u)\right|\leqslant C\gamma_{2n}^{\delta/2},$$

and

$$J_{3n}=\sup_u\left|\Phi(u/\sqrt{B_n})-\Phi(u)\right|\leqslant\left|s_n^2-1\right|\leqslant C\left(\gamma_{1n}^{1/2}+\gamma_{2n}^{1/2}+u(q)\right),$$

therefore, which together with Lemmas 5 and 10, taking $\delta=1$ and noting that $\gamma_{2n}^{1/2}\leqslant\gamma_{2n}^{1/3}$, we have

$$\sup_u|P(S_n\leqslant u)-\Phi(u)|\leqslant\left(\gamma_{1n}^{1/3}+\gamma_{2n}^{1/3}+\gamma_{2n}^{\delta/2}+u^{1/3}(q)+v(n)u^{-2/3}(q)+P(|S''_n|>\gamma_{1n}^{1/3})+P(|S'''_n|>\gamma_{2n}^{1/3})\right)$$
$$\leqslant C\left(\gamma_{1n}^{1/3}+\gamma_{2n}^{1/3}+u^{1/3}(q)+v(n)u^{-2/3}(q)\right).$$

The proof is completed.

Proof of Corollary 1 In Theorem 1 let $p=\lfloor n^{2/3}\rfloor$, $q=\lfloor n^{1/3}\rfloor$, $b_n=O(n^{-1/3})$, $\rho^-(n)=O(n^{-2})$. It is easy to verify

$$\gamma_{1n}^{1/3}+\gamma_{2n}^{1/3}=O(n^{-1/9}),\ \ u^{1/3}(q)\leqslant C\left[\sum_{t=q}^{\infty}t^{-2}\right]^{1/3}=O(n^{-1/9}).$$

Moreover, for some $r>2$, by $b_n=O(n^{-1/3})$ we can obtain $v(n)u^{-2/3}(q)=O(n^{-1/9})$. The proof is completed.

Proof of Theorem 2 In order to complete the proof, it suffices to verify by Theorem 1,

$$\frac{|E\hat{f}_n(x)-f(x)|}{\sqrt{\mathrm{Var}(\hat{f}_n(x))}}=O\left((nb_n^3)^{1/2}\right). \tag{18}$$

Since $f(x)$ satisfies the local Lipschitz condition, i.e. for all $(k_{-1}-1/2)b_n\leqslant x,y\leqslant(k_1+1/2)b_n$ and for some $L>0$, $|f(x)-f(y)|\leqslant L|x-y|\leqslant Cb_n$. Noting that $0<1/2+k_0-x/b_n\leqslant 1$ and $0<1/2-k_0+x/b_n\leqslant 1$, it follows from (1) that

$$\begin{aligned}\left|E\hat{f}_n(x)-f(x)\right|&=\left|\frac{1}{2}\left(\frac{1}{2}+k_0-\frac{x}{b_n}\right)(nb_n)^{-1}\left(\sum_{i=1}^{n}P\left((k_{-1}-1/2)b_n\leqslant X_i<(k_{-1}+1/2)b_n\right)+\sum_{i=1}^{n}P\left((k_0-1/2)b_n\leqslant X_i<(k_0+1/2)b_n\right)\right)\right.\\&\quad\left.+\frac{1}{2}\left(\frac{1}{2}-k_0+\frac{x}{b_n}\right)(nb_n)^{-1}\left(\sum_{i=1}^{n}P\left((k_0-1/2)b_n\leqslant X_i<(k_0+1/2)b_n\right)+\sum_{i=1}^{n}P\left((k_1-1/2)b_n\leqslant X_i<(k_1+1/2)b_n\right)\right)-f(x)\right|\\&=\left|\frac{1}{2}\left(\frac{1}{2}+k_0-\frac{x}{b_n}\right)b_n^{-1}\left(P((k_{-1}-1/2)b_n\leqslant X<(k_{-1}+1/2)b_n)+P((k_0-1/2)b_n\leqslant X<(k_0+1/2)b_n)\right)\right.\\&\quad\left.+\frac{1}{2}\left(\frac{1}{2}-k_0+\frac{x}{b_n}\right)b_n^{-1}\left(P((k_0-1/2)b_n\leqslant X<(k_0+1/2)b_n)+P((k_1-1/2)b_n\leqslant X<(k_1+1/2)b_n)\right)-f(x)\right|\\&\leqslant b_n^{-1}\int_{(k_{-1}-1/2)b_n}^{(k_1+1/2)b_n}|f(y)-f(x)|\mathrm{d}y\leqslant Cb_n.\end{aligned} \tag{19}$$

On the other hand, by Lemma 4 we can obtain that for all n large enough,

$$\mathrm{Var}\left(\hat{f}_n(x)\right)\geqslant f(x)/(4nb_n),$$

which together with (19) derives (18). Therefore, we complete the proof of Theorem 2.

Proof of Corollary 2 Similar to the proof of Corollary 1, take $b_n=O\left(n^{-\alpha}\right)$, $11/27\leqslant\alpha<2/3$, let $p=\lfloor n^{2/3}\rfloor$, $q=\lfloor n^{1/3}\rfloor$, $\rho^-(n)=O(n^{-2})$. It is easy to verify that

$$\gamma_{1n}^{1/3}+\gamma_{2n}^{1/3}+u(q)^{1/3}+(nb_n^3)^{1/2}=O\left(n^{-1/9}\right),$$

moreover, for some $r>2$, we can also obtain $v(n)u^{-2/3}(q)=O\left(n^{-1/9}\right)$. The result follows immediately from Theorem 2 and the above. The proof is completed.

Proof of Corollary 3 The result follows directly by letting each error term tend to zero in Theorem 2.

3 Numerical Simulation

In this section, we perform numerical simulations to investigate the finite sample performance of the edge frequency polygon estimator based on ANA random variables under the following two cases. The random numbers are generated as follows.

Case 1 For any fixed $n\geqslant 3$, let normal random vector $\left(X_1,X_2,\cdots,X_n\right)\sim N_n(\mathbf{0},\boldsymbol{\Sigma})$, where $\mathbf{0}$ represents zero vector and

$$\boldsymbol{\Sigma}=\begin{pmatrix}1+\theta^2&-\theta&0&\cdots&0&0&0\\-\theta&1+\theta^2&-\theta&\cdots&0&0&0\\0&-\theta&1+\theta^2&\cdots&0&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&0&\cdots&1+\theta^2&-\theta&0\\0&0&0&\cdots&-\theta&1+\theta^2&-\theta\\0&0&0&\cdots&0&-\theta&1+\theta^2\end{pmatrix}_{n\times n},$$

where $0<\theta<1$, $X_i\sim N(0,1+\theta^2)$, $i\geqslant 1$. From Joag-Dev and Proschan[23], it can be seen that $\left(X_1,X_2,\cdots,X_n\right)$ is an NA

vector. For some fixed integer $m \geqslant 1$, let $Y_i = \sum_{k=0}^{m} X_{i+k}$ for each $i \geqslant 1$, then it is easy to show that $\{Y_i,\ i \geqslant 1\}$ is a sequence of ρ^*-mixing random variables and thus a sequence of ρ^--mixing random variables with $EY_i = 0$.

For the purpose of comparison of the different estimators, we consider the edge frequency polygon estimator $\hat{f}_n(x)$, the frequency polygon estimator $\widetilde{f}_n(x)$ and the Epanechnikov kernel estimator (i. e., the kernel $K(u) = 0.75(1 - u^2)I(|u| \leqslant 1)$). Take $m = 2$, $\theta = 0.2$, 0.5 and the sample size n as $n = 200$, 400, 600, respectively. The optimal bandwidth for the Epanechnikov kernel estimator is chosen by cross validation (CV) method. We repeat the calculation of 1 000 times using the R software to obtain the mean square error (MSE) of the estimators for different sample sizes and $x =$ 0.5, 0.75, respectively. As shown in Table 1, for some fixed x and θ, it can be observed that the MSE of the estimators decrease significantly as the sample size n increases. Moreover, the MSE of the estimator $\hat{f}_n(x)$ is consistently smaller than the corresponding MSE of $\widetilde{f}_n(x)$, however, there is no significant difference between the edge frequency polygon estimator and the Epanechnikov kernel estimator for a fixed sample size.

Specifically, we also generate the Quantile-Quantile (Q-Q in short) plots of $\widetilde{S}_n$, which assess the asymptotic normality of the estimators in Figs. 1 and 2. The distribution of $\widetilde{S}_n$ should be asymptotically normal according to Theorem 2. Figs. 1 and 2 are the Q-Q plots of $\widetilde{S}_n$ with $\theta = 0.2$, $\theta = 0.5$ and sample size $n = 200$, 400, 600, respectively. From Figs. 1 and 2, it can be seen that $\widetilde{S}_n$ tends to the standard normal distribution as n. These results fit well with the theoretical results we derived in the previous section, further confirming the validity of the proposed estimators.

Case 2 Let $\epsilon_i \overset{\text{i.i.d.}}{\sim} N(0, 3/4)$, $i \geqslant 1$. Set $Z_1 \sim N(0, 1)$, and $Z_i = 0.5Z_{i-1} + \epsilon_i$ for each $i \geqslant 2$. Then $\{Z_i, i \geqslant 1\}$ is a sequence of ρ^--mixing random variables with $Z_i \sim N(0, 1)$. For simplicity, the other settings are the same as those in Case 1.

As shown in Table 2 for some fixed x, we can also obtain the similar conclusions as those in Case 1. From Figs. 3 and 4, it can also be seen that $\widetilde{S}_n$ tends to the standard normal distribution as n. These results fit well with our main results established in Section 2.

Table 1 MSE of the different estimators with θ=0.2 and θ=0.5 in case 1

x	n	θ=0.2			θ=0.5		
		$\hat{f}_n(x)$	$\widetilde{f}_n(x)$	Epanechnikov kernel estimator	$\hat{f}_n(x)$	$\widetilde{f}_n(x)$	Epanechnikov kernel estimator
0.5	200	$1.534\,992\times10^{-3}$	$7.006\,038\times10^{-3}$	$0.870\,667\times10^{-3}$	$1.490\,180\times10^{-3}$	$7.757\,300\times10^{-3}$	$0.997\,438\times10^{-3}$
	400	$0.855\,714\times10^{-3}$	$4.243\,047\times10^{-3}$	$0.531\,311\times10^{-3}$	$0.886\,136\times10^{-3}$	$4.666\,696\times10^{-3}$	$0.620\,228\times10^{-3}$
	600	$0.569\,999\times10^{-3}$	$2.648\,557\times10^{-3}$	$0.390\,412\times10^{-3}$	$0.563\,563\times10^{-3}$	$3.371\,044\times10^{-3}$	$0.436\,002\times10^{-3}$
0.75	200	$1.173\,086\times10^{-3}$	$6.499\,233\times10^{-3}$	$0.761\,878\times10^{-3}$	$1.146\,780\times10^{-3}$	$7.704\,812\times10^{-3}$	$0.818\,208\times10^{-3}$
	400	$0.869\,209\times10^{-3}$	$3.816\,499\times10^{-3}$	$0.449\,555\times10^{-3}$	$0.911\,427\times10^{-3}$	$4.147\,423\times10^{-3}$	$0.507\,787\times10^{-3}$
	600	$0.566\,590\times10^{-3}$	$2.744\,496\times10^{-3}$	$0.323\,313\times10^{-3}$	$0.547\,834\times10^{-3}$	$3.504\,102\times10^{-3}$	$0.352\,368\times10^{-3}$

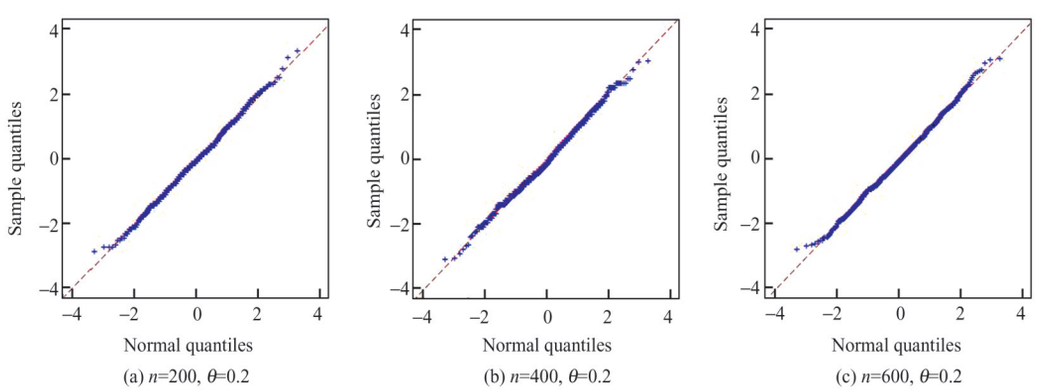


Fig. 1 Q-Q plots of $\widetilde{S}_n$ with n=200, 400, 600 under θ=0.2

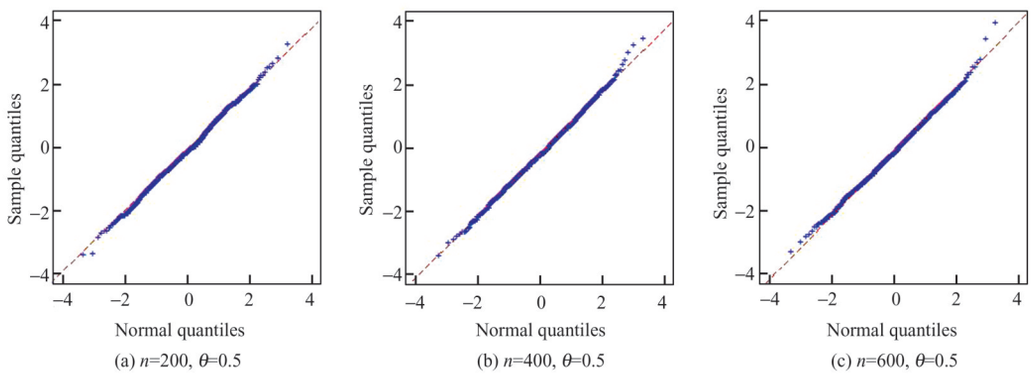


Fig. 2 Q-Q plots of $\widetilde{S}_n$ with n=200, 400, 600 under θ=0.5

Table 2 MSE of the different estimators in case 2

x	n	$\hat{f}_n(x)$	$\widetilde{f}_n(x)$	Epanechnikov kernel estimator
0.5	200	$2.193\,930\times10^{-3}$	$4.713\,003\times10^{-3}$	$1.954\,092\times10^{-3}$
	400	$1.221\,682\times10^{-3}$	$2.238\,906\times10^{-3}$	$1.017\,245\times10^{-3}$
	600	$0.800\,506\times10^{-3}$	$1.203\,738\times10^{-3}$	$0.804\,568\times10^{-3}$
0.75	200	$1.712\,701\times10^{-3}$	$2.370\,250\times10^{-3}$	$1.553\,220\times10^{-3}$
	400	$1.252\,575\times10^{-3}$	$2.422\,586\times10^{-3}$	$0.808\,627\times10^{-3}$
	600	$0.800\,031\times10^{-3}$	$1.325\,517\times10^{-3}$	$0.644\,946\times10^{-3}$

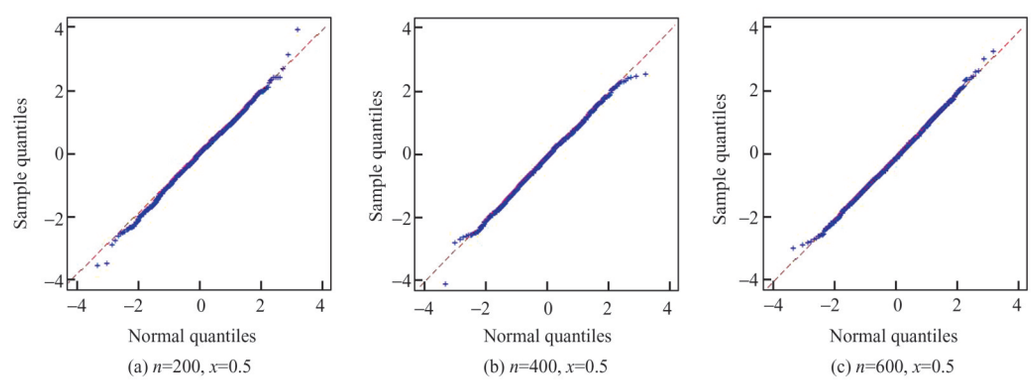


Fig. 3 Q-Q plots of $\widetilde{S}_n$ with n=200, 400, 600 under x=0.5

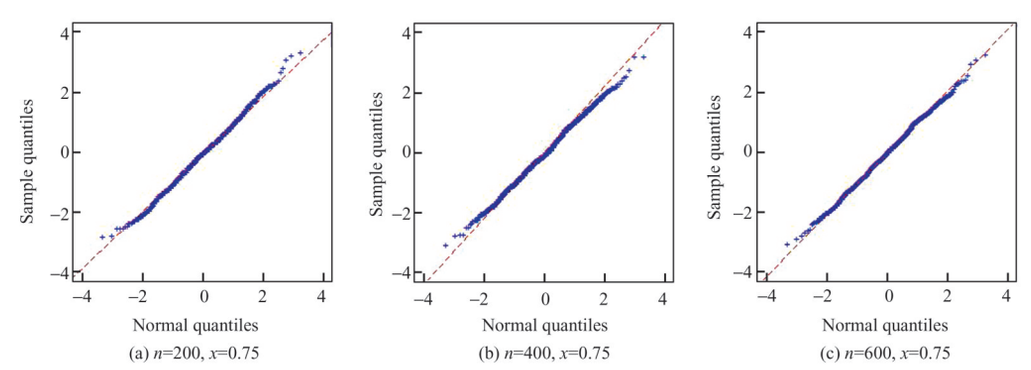


Fig. 4 Q-Q plots of $\widetilde{S}_n$ with n=200, 400, 600 under x=0.75

References

[1] Scott D W. Frequency polygons: Theory and application[J]. *Journal of the American Statistical Association*, 1985, **80** (390): 348-354.

[2] Jones M C, Samiuddin M, Al-Harbey A H, *et al*. The edge

frequency polygon[J]. *Biometrika*, 1998, **85**(1): 235-239.

[3] Carbon M, Francq C, Tran L T. Asymptotic normality of frequency polygons for random fields[J]. *Journal of Statistical Planning and Inference*, 2010, **140**(2): 502-514.

[4] Yang X. Frequency polygon estimation of density function for dependent samples[J]. *Journal of the Korean Statistical Society*, 2015, **44**(4): 530-537.

[5] Xing G D, Yang S C, Liang X. On the uniform consistency of frequency polygons for ψ-mixing samples[J]. *Journal of the Korean Statistical Society*, 2015, **44**(2): 179-186.

[6] Wen Y Z, Xing G D. On the uniformly asymptotic normality of frequency polygons for ψ-mixing samples[J]. *Communications in Statistics—Theory and Methods*, 2017, **46**(21): 10416-10425.

[7] Xing G D, Yang S C. Uniformly strong consistency of frequency polygons for negatively associated samples[J]. *Communications in Statistics—Simulation and Computation*, 2017, **46**(3): 2168-2175.

[8] Xing G D, Yang S C, Li X H. Uniformly strong consistency and Berry-Esséen bound of frequency polygons for α-mixing samples[J]. *Communications in Statistics—Simulation and Computation*, 2019, **48**(2): 416-430.

[9] Kang S L, Abbasi R, Wang H L, *et al*. Asymptotic behaviour of frequency polygons under φ-mixing samples[J]. *Communications in Statistics—Theory and Methods*, 2018, **47**(21): 5369-5377.

[10] Xi M M, Wang C H, Wang X J. Uniformly strong consistency and the rates of asymptotic normality for the edge frequency polygons[J]. *Statistics*, 2023, **57**(6): 1444-1468.

[11] Younso A. Consistency of the frequency polygon estimators of density mode for strongly mixing processes[J]. *Communications in Statistics—Theory and Methods*, 2023, **52**(7): 2182-2197.

[12] Wu Y, Wang X J. On Berry-Esséen bound of frequency polygon estimation under ρ-mixing samples[J]. *Metrika*, 2025, **88**(1): 19-41.

[13] Wang W, Wang H, Wu Y. Consistency of the frequency polygon estimators of density function and density mode under *m*-END samples[J]. *Communications in Statistics—Theory and Methods*, 2025, **54**(5): 1271-1286.

[14] Chao X L, Hu X P. Asymptotic normality of the frequency polygon estimators under widely orthant dependent samples[J]. *Communications in Statistics—Simulation and Computation*, 2026, **55**(9): 3700-3717.

[15] Bradley R C. On the spectral density and asymptotic normality of weakly dependent random fields[J]. *Journal of Theoretical Probability*, 1992, **5**(2): 355-373.

[16] Zhang L X, Wang X Y. Convergence rates in the strong laws of asymptotically negatively associated random fields[J]. *Applied Mathematics—A Journal of Chinese Universities*, *Series B*, 1999, **14**(4): 406-416.

[17] Zhang L X. Central limit theorems for asymptotically negatively associated random fields[J]. *Acta Mathematica Sinica*, 2000, **16**(4): 691-710.

[18] Tang X F, Xi M M, Wu Y, *et al*. Asymptotic normality of a wavelet estimator for asymptotically negatively associated errors[J]. *Statistics & Probability Letters*, 2018, **140**: 191-201.

[19] Wu Y, Wang X J, Shen A T. Strong convergence properties for weighted sums of *m*-asymptotic negatively associated random variables and statistical applications[J]. *Statistical Papers*, 2021, **62**(5): 2169-2194.

[20] Petrov V V. Limit theorems of probability theory: Sequences of independent random variables[J]. *Proceedings of the Edinburgh Mathematical Society*, 1996, **39**(3):591-592.

[21] Ledoux M, Talagrand M. *Probability in Banach Spaces*[M]. Berlin, Heidelberg: Springer-Verlag, 1991.

[22] Yang S C. Uniformly asymptotic normality of the regression weighted estimator for negatively associated samples[J]. *Statistics & Probability Letters*, 2003, **62**(2): 101-110.

[23] Joag-Dev K, Proschan F. Negative association of random variables with applications[J]. *The Annals of Statistics*, 1983, **11**(1): 286-295.

渐近负相依随机样本下边缘频率多边形估计的渐近正态性

胡学平， 王子予

安庆师范大学 数学与统计学院, 安徽 安庆 246133

摘要：基于直方图技术的边缘频率多边形估计，其收敛速度与非负核估计相当，且具有计算简便的优势。本文在合适的条件下，探讨了基于渐近负相依（ANA）随机序列的边缘频率多边形估计的一致渐近正态性，获得其收敛速度近似于$O\left(n^{-1/9}\right)$，表明了该估计具有与核估计类似的Berry-Esséen界。此外，通过数值模拟验证了上述理论结果。

关键词：一致渐近正态性；边缘频率多边形估计；渐近负相依随机变量

□